A Nonlinear Electrical Circuit Exhibiting Period Doubling Bifurcation and Chaotic Behavior

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By constructing a circuit consisting of a diode, resistor, and inductor in series with sinusoidally-driven function generator, period doubling bifurcation and chaotic behavior were observed in this experiment. By using a fixed frequency close to the value of resonance frequency of the circuit and adjusting the voltage amplitude of the function generator, the circuit exhibited chaotic behavior. A bifurcation diagram of the data was created at a fixed frequency of 376 kHz to illustrate the period doubling route to chaos that the circuit experienced as the function generator's voltage increased. Two trials where the voltage of the function generator and across the diode were recorded were taken at two different fixed frequencies: 276 kHz and 376 kHz. Determining where at least three period doubling bifurcation points occurred in each trial allowed for the calculations of the Feigenbaum constant, $\delta = 4.669$, for each trial. For the 276 kHz fixed frequency $\delta = 4.1 \pm 0.7$ and for the 376 kHz fixed frequency $\delta = 4.0 \pm 0.7$, which is 13% and 14% difference from the accepted value respectively.

I. INTRODUCTION

Chaos is a class of complex behaviors that can emerge from nonlinear dynamic systems and can be seen in both the natural world and technology. Mitchell Jay Feigenbaum, one of the early pioneers of chaos theory, published a paper in 1978 proving that if a system exhibits repeated period doubling by increasing some finite parameter, then the system will have an infinite number of period doubling bifurcations in a finite increase of that parameter [1]. Feigenbaum mathematically expressed this idea by the following recursive equation,

$$x_{n+1} = \lambda x_n (1 - x_n), \tag{1}$$

where x_n is a normalized population at generation n, and the normalized population exhibits an extraordinary range of subtle behavior as the parameter λ varies [2].

He also observed that a wide class of nonlinear systems are governed by universal constants. These constants have been observed in many physical systems including electronic circuits, called Feigenbaum constants. A simple electronic circuit which exhibits this period doubling route to chaos is the chaotic resonator, first demonstrated in 1981 by Paul Linsay [3]. The circuit is composed of three basic components in series: a resistor, inductor and diode. This is the circuit which will be used for analysis in this lab.

II. THEORY

A circuit with a resistor, inductor, and diode in series, a RLD circuit, exhibits chaotic behavior if the input driving voltage is periodic. Due to the non-linear response of the diode, the system will exhibit chaotic behavior. When a low voltage is applied against the forward-bias of the diode, the diode will act like a capacitor. The capacitance is due to the presence of the charge layers associated with the diode's pn junction. Applying a larger forward current to the circuit will cause an increase in the amount of charge that cross the junction and require a longer time for the system to return to its reserve bias equilibrium. If the reverse current is unable to reach equilibrium before the forward bias, then the next cycle will depend upon the previous cycle, which is equivalent to different parameters at each cycle's initial condition [4]. Thus, the varying conditions might cause chaos. In a RLD circuit, chaos may eventually result due to the unrecombined electrons and holes that cross the forward-biased pn junction and as the diode changes state of bias, diffuse back to their origin [4].

Since the diode has this capacitance like behavior, a frequency where chaos is possible can be expressed as the resonance frequency for the equivalent RLC series circuit [5],

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC_j}}, \tag{2}$$

where f_0 is the resonance frequency, L is inductance, and C_j is the capacitance of the diode. When a RLD circuit is driven by an low amplitude input signal with a frequency near or equal to its resonance frequency f_0, the circuit will operate normally as expected [6]. If the input signal amplitude is increased, the output signal periodic state will change and be divided into two frequency components dependent on f_0, which is a process known as period doubling bifurcation. Further amplitude increase will result in splitting each new component, leading progressively into higher periodicity until there are so many states that it appears as chaos.

Since the circuit driven by a sinusoidal signal, its output signal power spectrum will contain the fundamental input frequency and some high order harmonics due to the system's nonlinearities. If the same system is to progress towards chaos by period-doubling then additional frequency components, known as subharmonics and ultra-subharmonics, will appear [7].

A. Period Doubling Bifurcation

Feigenbaum expressed mathematically that the period doubling route to chaos occurs if the system exhibits period doubling by increasing a single parameter. He proved that the system which experienced this period doubling route to chaos would have a universal constant $\delta = 4.669...$ [2] such that,

$$\delta = \lim_{n \to +\infty} \frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+2} - \lambda_{n+1}}, \quad (3)$$

where λ_n is the parameter value at which the nth bifurcation occurred. Due to the fact that this experiment is trying to reproduce chaos with an electrical circuit, by holding a particular driving frequency constant and adjusting the voltage, Eqn. 3 will take the form,

$$\delta = \lim_{n \to +\infty} \frac{V_{n+1} - V_n}{V_{n+2} - V_{n+1}}, \quad (4)$$

where V_n is the voltage value of the driving frequency from a function generator at which the nth bifurcation occurred.

III. EXPERIMENT

This focus of this experiment was the observation of chaotic behavior in an RLD circuit. The diode used in this lab was a IN5401, the resistor had a resistance of 4.8 Ω, and the inductor had an inductance of 221 μH. The circuit diagram is shown in Fig. 1.

After the circuit was constructed, a function generator was used to drive a sinusoidal wave into the circuit at varying amplitudes. As this was done, a two-channel oscilloscope monitored the voltage across the function generator and the diode. As the circuit is driven with higher peak-to-peak amplitudes, the circuit takes on nonlinear behaviors. By observing the voltages of both channels in an X-Y display mode of the oscilloscope, one could tell at what driving voltage period doubling bifurcation would occur. Then by switching the oscilloscope back to a Y-T display mode, the observer would be able to visually see both monitored voltages and that the voltage across the diode had doubled its period (if it was the first bifurcation) relative to the input voltage.

Once bifurcation had been detected, by recording the peak-to-peak voltage of the function generator and the peaks of the voltage across the diode a bifurcation diagram of the circuit was constructed. Essentially each time a bifurcation occurred, the period of the diode became twice as long as the period of the driving voltage. Thus, an observer would have to record two peaks of the diode voltage after the first bifurcation, then four peaks of the diode voltage after the second bifurcation and so on. Plotting the driving voltage versus the newly created peaks in the voltage across the diode created a bifurcation diagram.

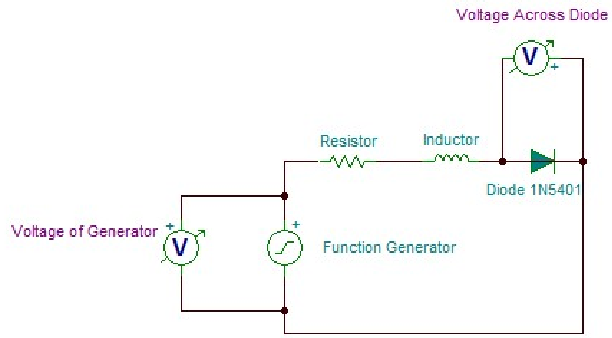

FIG. 1. This is the circuit diagram in this lab. A function generator was used to drive a sinusoidal wave into the circuit, which consisted of a resistor, inductor, and diode in series. The input voltage of the generator and voltage across the diode were measured by an oscilloscope as the amplitude of the input waves increased.

A thumb drive was used to transport data measured from the oscilloscope to the computer. Once on the computer, the data was analyzed using Igor Pro.

IV. DATA AND RESULTS

Two trials were taken at two different fixed frequencies: 376 kHz and 276 kHz. The trial which held the frequency fixed at a 376 kHz is the focus in this analysis because four period doubling bifurcations and chaotic behavior were observed. Since this fixed frequency had so many observable bifurcations it was an ideal candidate to create a bifurcation diagram. Fig. 4 shows the phase space plots, or X-T mode on the oscilloscope, and voltage versus time plots, or Y-T mode on the oscilloscope, of the voltage of the function generator versus the voltage across the diode for all measurable bifurcations. In phase space, period-2 bifurcation appears when a single peak splits into two separate peaks, as shown in image (a) in Fig. 4. Similarly, images (c), (e), and (g) in Fig. 4 are the phase space diagrams for period-4, period-8, and period-16 bifurcation respectively. In each progressing period doubling there are twice as many peaks in the wave form. After period-16 bifurcation, it becomes more difficult to distinguish peaks which have split from each other, until eventually a whole range of peaks is present and chaotic behavior is observed, as shown in Fig. 3.

Images (b), (d), (f), and (h) are the Y-T display data of both the voltage across the diode and the driving voltage of the function generator over a sampled time period for period-2, period-4, period-8, and period-16 bifurcation respectively. The boxed off areas represent where period doubling occurs and where the measurements of voltages to determine the bifurcation diagram were recorded.

Table I shows when period doubling bifurcations occur at certain fixed frequency and function generator volt-

TABLE I. The voltage recorded from the function generator at which period bifurcation happened at a fixed frequency.

	Voltage of Function Generator (V)			
Frequency of Function Generator (kHz)	Period-2 Bifurcation	Period-4 Bifurcation	Period-8 Bifurcation	Period-16 Bifurcation
376	1.00	1.32	1.41	1.43
276	0.530	0.615	0.636	N/A

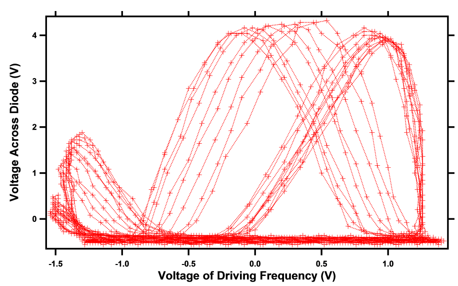

FIG. 2. Phase plot of chaos at a voltage above 1.48 V and fixed driving frequency of 376 kHz from the function generator. Notice the range of different peaks present, making it impossible to distinguish peaks from each other.

ages. From the 376 kHz data set, Feigenbaum's constant and uncertainty can be determined. Since Feigenbaum's constant only requires that three bifurcation points be necessary to calculate δ, two Feigenbaum constants can be calculated from the 376 kHz data set. For the period-2 through period-8 bifurcation points $\delta = 3.6$ and for the period-4 through period-16 bifurcation points $\delta = 4.5$. Given these two values a standard deviation or uncertainty and average can be determined for δ. Thus, for 376 kHz data set $\delta = 4.0 \pm 0.7$. Although the 276 kHz data set had fewer period doubling bifurcations than the 376 kHz data set before turning into chaos, it contained enough bifurcations points to determine Feigenbaum's constant. Thus, for the 276 kHz data set $\delta = 4.1 \pm 0.7$.

A bifurcation diagram was created from the phase plot

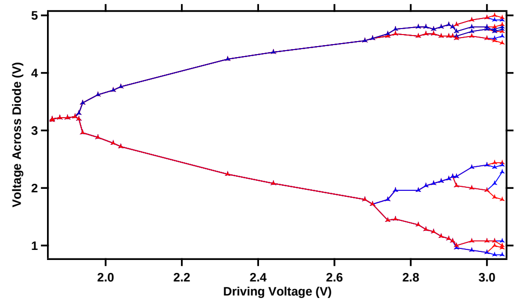

FIG. 3. Phase plot of voltage across diode and driving voltage at fixed frequency of 376 kHz. This is also called a bifurcation diagram, and it exhibits a period doubling route to chaos. A line splits into two new lines when a bifurcation occurred. Period-16 bifurcations are shown here.

of voltages recorded across the diode and the driving voltage of the function generator as bifurcations occurred, as seen in Fig. 3. The bifurcation diagram shows the period doubling route to chaos taken by the system as the voltage amplitude increases. Four bifurcations have occurred, which resulted in a period-16 behavior in the voltage across the diode.

V. CONCLUSION

The purpose of this experiment was to construct a simple circuit with a resistor, diode and inductor in series driven by sinusoidal signal to create chaotic behavior within the circuit. Due to the nonlinear dynamics of a diode, when an RLD circuit is created and driven by sinusoidal signal, whose amplitude is continually increased, chaotic behavior can be observed. The circuit constructed exhibited period-doubling bifurcation route to chaos for two separate fixed frequencies, 276 and 376 kHz. Phase plot diagrams were created from the 376 kHz data set to exhibit what period doubling bifurcation appeared as on the oscilloscope. The circuit also verified the Feigenbaum universal constant. For the 276 kHz fixed frequency $\delta = 4.1 \pm 0.7$ and for the 376 kHz fixed frequency $\delta = 4.0 \pm 0.7$, which is 13% and 14% difference from the $\delta = 4.669$ accepted value respectively. However, taking into account the uncertainties of each of the experimental data sets, each value falls within range of the accepted value.

The bifurcation diagram shows that, just as Feigenbaum predicted that once bifurcation occurs, it continues to occur at an increasing rate, which are seen as smaller difference in voltage of the driving voltage between each bifurcation point. The bifurcation diagram also illustrates how period doubling routes to chaotic behavior in the system.

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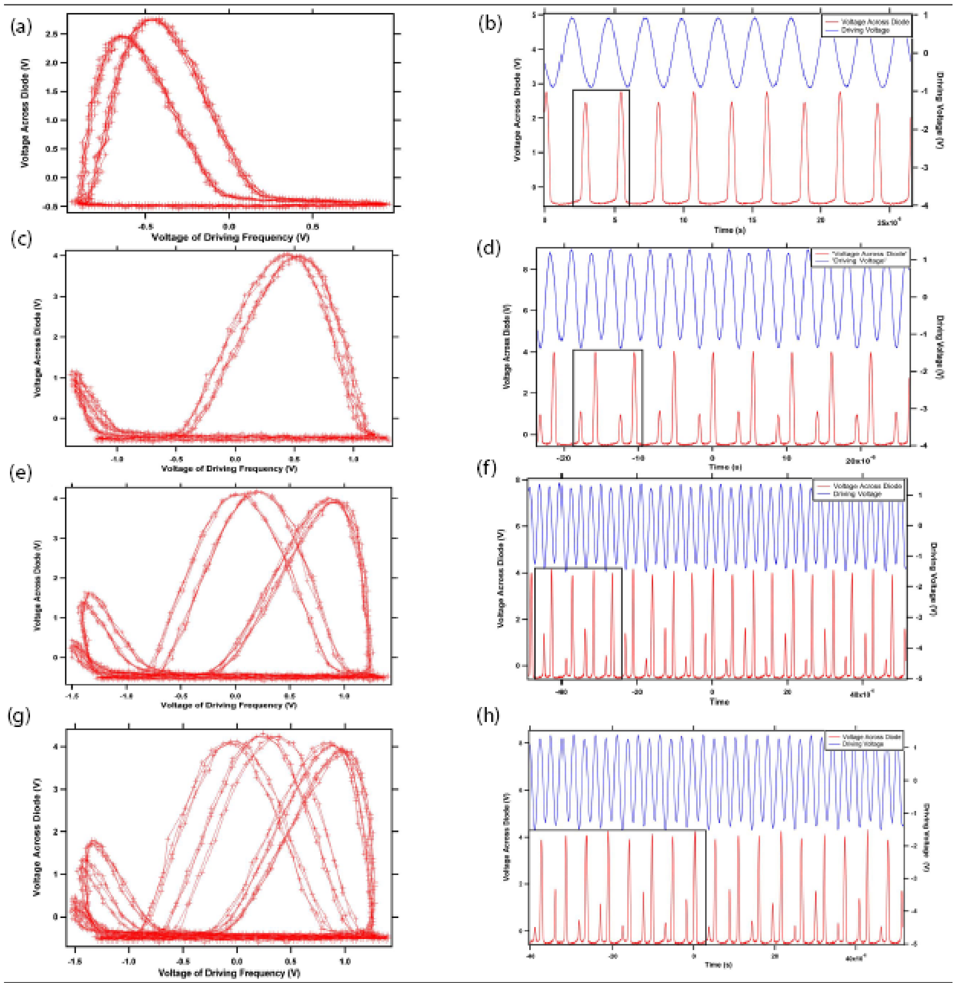

FIG. 4. Above are plots of various period bifurcations, which occurred at the fixed frequency of 376 kHz but with different driving voltages. The left column represent the X-Y mode of the oscilloscope and the right column represent the Y-T mode of the oscilloscope. The X-Y mode is a graph of the voltage of the diode versus the driving voltage. The Y-T mode is a graph of the voltage of the diode and driving voltage versus time. Boxed regions in the Y-T mode show where period doubling occurs for the voltage across the diode. Period-2 bifurcation can be seen in images (a) and (b). Period-4 bifurcation can be seen in images (c) and (d). Period-8 bifurcation can be seen in images (e) and (f). Period-16 bifurcation can be seen in images (g) and (h).

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